Low Basis Theorem and Model Theory without Actual Infinity

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XVII(I) Logic Workshop
Chęciny, 15–17 September 2016

Outline

Plan

Introduction

- We investigate model theoretic-constructions

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- We study how these constructions can be performed without actual infinity

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- We study how these constructions can be performed without actual infinity
- We use the notion of FM-representability as an explication of "expressibility without actual infinity"

Plan

Concrete Models

Definition

Let $\sigma = \{P_1, \ldots, P_k, C\}$ be a finite relational (concrete) vocabulary.
We say that $\mathcal{A} = (\varphi_U, \varphi_{P_1}, \ldots, \varphi_{P_k}, \varphi_C, \varphi_{\models})$ is a concrete σ-model when:

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- $\varphi_\models$ FM-represents the satisfaction relation on $(U, R_1, \ldots, R_k, C^I)$

Concrete Models 2

Concrete model-theoretic notions

- Expansion, Reduction

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Concrete model-theoretic notions

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- Chains of models

Concrete Models 3

Problems

- Images under concrete morphisms and diagrams

Concrete Models 3

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- Sums of arbitrary chains

Concrete Models 3

Problems

- Images under concrete morphisms and diagrams
- Sums of arbitrary chains
- Glueing models (to be explained later)

Concrete Completeness

Theorem

Let T be a consistent theory such that $\mathrm{Cn}(T)$ is concrete. Then there is a concrete model $\mathcal{A}$ of T.

Plan

Robinson's Construction

Craig's Interpolation Lemma

Let φ be a L_1-sentence, let ψ be a L_2 sentence and let $\varphi \models \psi$. Then there is a $L_1 \cap L_2$ sentence θ such that $\varphi \models \theta$ and $\theta \models \psi$.

Robinson's Construction

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Separability

Let T_1 be a theory in language L_1 and let T_2 be a theory in language L_2. We say that T_1 and T_2 are separable in there is an $L_1 \cap L_2$ sentence θ such that $T_1 \vdash \theta$ and $T_2 \vdash \neg\theta$.

Robinson's Construction 2

Proof of Craig's Interpolation Lemma

Let φ and ψ be as in the statement of the theorem. Suppose for the sake of contradiction that there is no interpolant.

- Construct a complete in $L_1 \cap L_2$ theory A such that $A \cup \{\varphi\}$ and $A \cup \{\neg\psi\}$ are inseparable.

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    - $T_0 = A \cup \{\varphi\}$,
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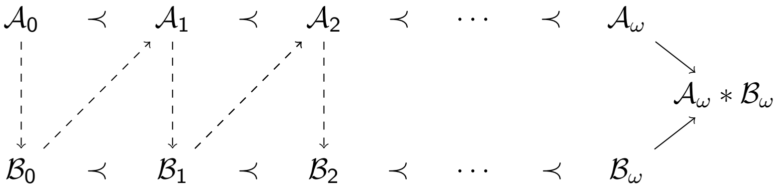
Robinson's Construction 3



Plan

Robinson's Construction in Concrete Framework

Proof of Craig's Interpolation Lemma

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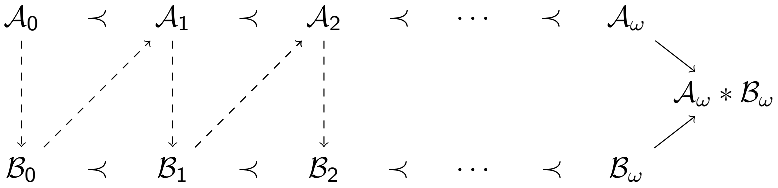
Robinson's Construction in Concrete Framework 2



Concrete completeness theorem is no sufficient!
We need a more convenient version of completeness theorem which we can interate.

Computations

- Turing machines with oracles

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- Turing machines with two oracles

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Computations

- Turing machines with oracles
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- Low sets
- Conventions
  $\sigma \in 2^{<\omega}, g \in 2^{\omega}, \Phi_i^{g}(n), \Phi_i^{\sigma \oplus g}(n), \sigma \in T, g \in [T], g \upharpoonright (i+1)$

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Corollary - Low Completeness Theorem

Let T be a consistent low theory. Then there is a low concrete model $\mathcal{A}$ such that $\mathcal{A} \models T$ and $\mathcal{A} \oplus T$ is low.

Low Basis Theorem 2

$$U_n = \{\sigma \in 2^{<\omega} \colon \Phi_n^{\sigma \oplus \mathcal{T}}(n) \uparrow\}.$$

Now let us inductively define a descending sequence of trees as follows.

$$T_0 = \mathcal{T},$$

$$T_{n+1} = \begin{cases} T_n & \text{if } T_n \cap U_n \text{ is finite} \\ T_n \cap U_n & \text{else} \end{cases}.$$

Low Basis Theorem 3

- Tricky algorithms – M_i

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- $f(i) = 1$ if and only if $T_{\ulcorner M_{2i} \urcorner + 1} \cap U_{\ulcorner M_{2i} \urcorner + 1}$

Low Basis Theorem 3

- Tricky algorithms – M_i
- $f(i) = 1$ if and only if $T_{\ulcorner M_{2i} \urcorner + 1} \cap U_{\ulcorner M_{2i} \urcorner + 1}$
- $IsFinite(T) \equiv \exists n \forall \sigma \, (\mathrm{lh}(\sigma) = n \Rightarrow \sigma \notin T)$

Low Basis Theorem 4

Lemma 1

Let $n \in \omega$. Then T_n is infinite and recursive in $\mathcal{T}$, thus low.

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Lemma 2

Let $i \in \omega$ and let $g \in 2^\omega$. Then:

$$g(i) = 1 \text{ if and only if } \Phi^g_{\ulcorner M_i \urcorner}(\ulcorner M_i \urcorner) \downarrow$$

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Lemma 3

Let $i \in \omega$. Then:

- $T_i \cap U_i$ is finite $\Rightarrow \forall g \in [T_{i+1}]\, \Phi_i^{g \oplus \mathcal{T}}(i) \downarrow$
- $T_i \cap U_i$ is infinite $\Rightarrow \forall g \in [T_{i+1}]\, \Phi_i^{g \oplus \mathcal{T}}(i) \uparrow$

Low Basis Theorem 5

Lemma 4

Let $i \in \omega$. Then the following are equivalent:

1. $T_{\ulcorner M_i \urcorner + 1} \cap U_{\ulcorner M_i \urcorner + 1}$ is finite
2. $\forall g \in [T_{\ulcorner M_i \urcorner + 1}]\, \Phi^{g \oplus \mathcal{T}}_{\ulcorner M_i \urcorner}(\ulcorner M_i \urcorner) \downarrow$
3. $\forall g \in [T_{\ulcorner M_i \urcorner + 1}]\, (g \oplus \mathcal{T})(i) = 1$

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Lemma 5

Let $i \in \omega$ and let $g \in [T_{\ulcorner M_{2i} \urcorner + 1}]$. Then

$$g \restriction (i+1) = f \restriction (i+1)$$

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Lemma 6

For every $k \in \omega$ it holds that $f \in [T_k]$.

Low Basis Theorem 6

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Let $i \in \omega$. Then the following are equivalent:

1. $T_i \cap U_i$ is finite
2. $\forall g \in [T_{i+1}]\Phi_i^{g \oplus \mathcal{T}}(i) \downarrow$
3. $\Phi_i^{f \oplus \mathcal{T}}(i) \downarrow$

Robinson's Construction in Concrete Framework - again

Proof of Craig's Interpolation Lemma

Let φ and ψ be as in the statement of the theorem.

- Construct a complete in $L_1 \cap L_2$ theory A such that $A \cup \{\varphi\}$ and $A \cup \{\neg\psi\}$ are inseparable. But how??
-
  - $T_0 = A \cup \{\varphi\}$,
  - $\mathcal{A}_i$ by low completeness from T_i
  - $S_0 = (\mathsf{ElDiag}(\mathcal{A}_0) \cap \mathrm{Sent}_{L_1 \cap L_2}) \cup \{\neg\psi\}$,
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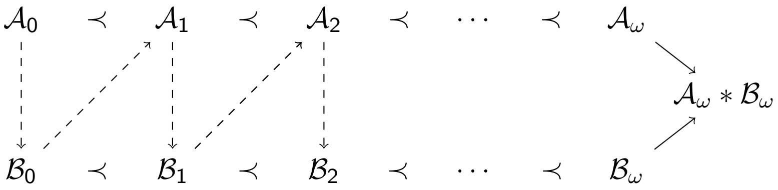
Robinson's Construction in Concrete Framework - again 2



Everything works... except for glueing.