The Δ^0_2 Turing Degrees: structure, definability and Cooper's question

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Turing Degrees

Definition

- For A and B contained in $\mathbb{N}$, A is *Turing reducible* to B ($B \geq_T A$) iff membership in A is computable when given membership information about B as data.
- A and B are *Turing equivalent* iff $A \geq_T B$ and $B \geq_T A$.
- The *Turing degrees* are the $\equiv_T$-equivalence classes. $\mathcal{D}$ denotes their partial ordering under $\geq_T$.

Two Lines of Study

Global: Properties of $\mathcal{D}$

- Elementary theory, definable relations, substructures, automorphisms.

Local: Properties of the Turing degrees of classes definable sets

- Recursively enumerable, recursively approximable, arithmetic, hyper-arithmetic.

Global Structure of the Turing Degrees

Existential Theory

Technical Fact

- A family of mutually Cohen generic subsets of $\mathbb{N}$ yield an independent collection of Turing degrees.
- Further, genericity over the elements of a set S yield a collection of Turing degrees independent over the degrees in that set.

Theorem (Kleene and Post, 1954)

Every countable partial order can be embedded as a sub-ordering of $\mathcal{D}$.

It follows that the $\exists$-theory of $\mathcal{D}$ is decidable.

Technical Fact (Spector, 1956)

Forcing with recursive perfect closed sets yields a set M of minimal Turing degree:

$$(\forall X)[M \geq_T X \Rightarrow (\emptyset \geq_T X \text{ or } X \geq_T M)].$$

Theorem (Lerman, 1971)

Every finite lattice can be embedded into $\mathcal{D}$ as initial segment.

It follows that the $\exists\forall$-theory of $\mathcal{D}$ is decidable.

Global Structure of the Turing Degrees

Elementary Theory

Theorem (Slaman and Woodin, 1986)

Every countable relation is definable from parameters in $\mathcal{D}$, uniformly in the number of arguments of the relation.

Theorem (Simpson, 1977)

The first order theory of $\mathcal{D}$ is recursively isomorphic to the second order theory of arithmetic.

Technical Fact

For A any countable antichain in $\mathcal{D}$ with upper bound b there are c_1 and c_2 such that A is the set of x which are minimal with respect to the following conditions.

- $b \geq_T x$
- $(\exists g)\big[c_1 \oplus x \geq_T g \;\;\&\;\; c_2 \oplus x \geq_T g \;\;\&\;\; x \not\geq_T g\big]$.

Global Structure of the Turing Degrees

Conjecture (Slaman and Woodin, 1990)

$\mathcal{D}$ is bi-interpretable with Second Order Arithmetic. That is, the relation "$\overrightarrow{p}$ codes $X \subseteq \mathbb{N}$ and x is the Turing degree of X" is definable in $\mathcal{D}$.

Theorem (Slaman and Woodin, 1990)

- *Bi-interpretability relative to parameters.*
- *Countable automorphism group for $\mathcal{D}$.*
- *Bi-interpretability if and only if $\mathcal{D}$ is rigid.*

Local Structure of the Turing Degrees

Gödel's Theorem tells us that axiomatic theories, despite modelling limited data generating environments, are not powerful enough to fully reflect the way in which knowledge is accumulated in real life. . . . Any attempt to transcend these limitations inevitably leads to a process of effective approximation, and the approximating complete theory is of Δ^0_2 *degree. —SBC*

S. Barry Cooper. Local degree theory. In, *Handbook of computability theory*. Volume 140, in Stud. Logic Found. Math. Pages 121–153. North-Holland, Amsterdam, 1999.

Local Structure of the Turing Degrees

The Turing degrees of the Δ_2^0 Sets: $\mathcal{D}(\leq_T 0')$

Remark

- *$0'$ is the Turing degree of the canonical example of a recursively enumerable, and thereby Δ_2^0, set which is not recursive.*
- *For $A \subseteq \mathbb{N}$ the following are equivalent.*
  - *A is recursively approximated.*
  - *A is Δ_2^0*
  - *A is recursive in $0'$.*

Theorem (Kleene and Post, 1954)

Every countable partial order can be embedded as a sub-ordering of $\mathcal{D}(\leq_T 0')$.

It follows that the $\exists$-theory of $\mathcal{D}(\leq_T 0')$ is decidable.

Local Structure of the Turing Degrees

Existential-Universal Theory

There are similarities between $\mathcal{D}$ and $\mathcal{D}(\leq_T 0')$.

Technical Facts

- (Sacks, 1961) There is a Δ^0_2-set of minimal Turing degree.
- (Lerman, 1983) Every finite lattice can be embedded into $\mathcal{D}(\leq_T 0')$ as an initial segment.

However, there is an obvious difference.

Technical Facts

- $0'$ is the greatest element of $\mathcal{D}(\leq_T 0')$.
- (Cooper, 1978) There are two minimal degrees m_1 and m_2 such that $m_1 \oplus m_2 \equiv_T 0'$.

Theorem (Lerman and Shore, 1988)

The $\exists\forall$-theory of $\mathcal{D}(\leq_T 0')$ is decidable.

- Combine Lerman's embedding of finite lattices with Cooper's join theorem: Every finite lattice can be embedded into $\mathcal{D}(\leq_T 0')$ as $\mathcal{L}$ with top element $0'$ so that $\mathcal{L} \setminus \{0'\}$ is an initial segment of $\mathcal{D}(\leq_T 0')$.
- Modify the decision procedure for $\mathcal{D}$ to account for the greatest element of $\mathcal{D}$ and its join reducibility to pairs of maximal elements of an embedded lattice.

Local Structure of the Turing Degrees

Elementary Theory

Theorem (Shore, 1981)

The first order theory of $\mathcal{D}(\leq_T 0')$ is recursively isomorphic to the first order theory of arithmetic.

Not the original proof

- The Slaman-Woodin coding of countable relations by parameters applies in $\mathcal{D}(\leq_T 0')$ to uniformly low antichains, and this allows for the representation of countable models of $P^- + I\Sigma_1$.
- Shore's technique to recognize the codings of the standard models.

Local Structure of the Turing Degrees

Cooper's Problem

There are two sorts of results:

- Structural characterizations in terms that are native to partial orderings.
- Logical characterizations in terms that correlate degree structures with models of arithmetic.

Of which sort is the following?

Problem (Cooper, 1999, Question 4.4)

Characterize the x-independent theory of $([x, x'], \leq_T)$.

Theorem

The set

$$\left\{ \varphi : (\forall x)\left[\ ([x, x'], \leq_T) \models \varphi\ \right] \right\}$$

is Π^1_1*-complete.*

We show there is a recursive function which maps Σ^1_1 sentences $\theta = (\exists W)(\forall n)\varphi(n, W \upharpoonright n)$ to a first order sentences θ^* so that

$$\theta \Leftrightarrow (\exists x)\big[\, ([x, x'], \leq_T) \models \theta^* \,\big].$$

- θ^* asserts "There are parameters $\overrightarrow{p}$ which code a model (M, W) with $(M, W) \models P^- + I\Sigma_1 + (\forall n)\varphi(n, W \upharpoonright n)$."
- Verify:
  - θ^* is a first order sentence.
  - If $(\exists x)\big[\, ([x, x'], \leq_T) \models \theta^* \,\big]$ then the standard part of the coded model exhibits a witness to $(\exists W)(\forall n)\varphi(n, W \upharpoonright n)$.
  - If $(\exists W)(\forall n)\varphi(n, W \upharpoonright n)$, then any x which computes such a W will code of $(\mathbb{N}, W)$ and so verify $(\exists x)\big[\, ([x, x'], \leq_T) \models \theta^* \,\big]$.

Problem

Characterize the theory of $([x, x'], \leq_T)$ on a cone.

Conjecture (Slaman and Woodin, 1990)

$\mathcal{D}(\leq_T 0')$ is bi-interpretable with First Order Arithmetic. That is, the relation "$\overrightarrow{p}$ codes $\mathbb{N}$ with element e and x is the Turing degree of the eth Δ^0_2 set" is definable in $\mathcal{D}(\leq_T 0')$.

Theorem (Slaman and Soskova, 2015)

- $\mathcal{D}(\leq_T 0')$ *has a finite automorphism base, and hence a countable automorphism group.*
  - *In fact, every automorphism of* $\mathcal{D}(\leq_T 0')$ *is arithmetically definable.*
- $\mathcal{D}(\leq_T 0')$ *is rigid iff bi-interpretable with First Order Arithmetic.*
- $\mathcal{D}(\leq_T 0')$ *is an atomic structure.*

Preliminary ingredients:

- Define an *indexing* of a set $\mathcal{Z} \subseteq \mathcal{D}(\leq_T 0')$ to be a function f from a copy of $\mathbb{N}$ onto $\mathcal{Z}$ such that f and the copy of $\mathbb{N}$ are both definable from parameters in $\mathcal{D}(\leq_T 0')$.
  - Note, if there is an indexing of $\mathcal{Z}$ then every arithmetically definable subset of $\mathcal{Z}$ is definable from parameters in $\mathcal{D}(\leq_T 0')$.
- (Slaman and Woodin, 1986) As previously mentioned, any uniformly low $\mathcal{Z}$ has an indexing; the recursively enumerable degrees have an indexing.
- (Following Jockusch and Posner, 1981) Every element of $\mathcal{D}(\leq_T 0')$ can be written as the meet of joins of low degrees.

Technical Fact

There is a uniformly low set of Turing degrees $\mathcal{Z}$, bounded by a low degree z, such that if $x, y <_T 0'$, $x' = 0'$ and $y \not\leq_T x$ then for $i \in \{1, 2\}$ there are recursively enumerable a_i and $c_i, b_i, g_i <_T 0'$ as follows.

- b_i and c_i are elements of $\mathcal{Z}$.
- g_i is the least element below a_i which joins b_i above c_i.
- $x \leq_T g_1 \oplus g_2$.
- $y \not\leq_T g_1 \oplus g_2$.

Every low degree x is uniquely determined by its interactions with the elements of $\mathcal{Z}$ and the recursively enumerable degrees. This gives an indexing of the set of x so determined, and hence an indexing of $\mathcal{D}(\leq_T 0')$.

Bi-interpretability and Automorphisms

Remaining Challenges

- Is $\mathcal{D}$ rigid?
- Is $\mathcal{D}(\leq_T 0')$ rigid?